Computing Conformal Maps onto Canonical Slit Domains

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Abstract. We extend the results of [2] by computing conformal maps onto the canonical slit domains in Nehari [14]. Along the way, we demonstrate the computability of solutions to Neuman problems.

1 Introduction

Let $\hat{\mathbb{C}}$ denote the extended complex plane. A *domain* is an open connected subset of $\hat{\mathbb{C}}$. A domain is *degenerate* if a component of its complement consists of a single point. A domain is *n-connected* if its complement has precisely n connected components and *finitely connected* if it is n-connected for some n.

In studying conformal mappings between domains in the extended complex plane it is convenient for both theoretical and practical purposes to introduce the so-called *canonical* domains and to study conformal maps of arbitrary domains onto these canonical domains. If the domain is 1-connected and non-degenerate, the canonical domain is the unit disk. In the case of doubly connected non-degenerate domains, the canonical domain is the annulus $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$. The *modulus* of this annulus is r_2/r_1. It is well-known that annuli with different moduli are not conformally equivalent (see, *e.g.*, [14] p. 333). When considering conformal mappings of domains with connectivity $n \geq 3$, it is convenient to consider canonical domains with different geometric characteristics.

Paul Koebe [11] outlined an iteration method for finding the conformal mapping from an n-connected domain to a circular domain (a domain whose complement consists of n disjoint closed disks). The convergence of his method was proved by Gaier [6], and the computability by Andreev, Daniel, and McNicholl [2]. The circular domains are the canonical domains in the recent constructions of the Schwarz–Christoffel mappings for domains that are sufficiently separated (see [3] and [5] and the refernces therein) and have been used as canonical domains in aircraft engineering as early as 1928 [1] and later by Halsey [8]. For

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numerous applications to nonlinear problems in mechanics see the monograph [13]. An appealing property of the circular domains as canonical domains is that recently there have been found explicit formulas for the Green's function (and, hence, for the Bergman kernel) for circular domains [10] and for the modified Green's function [4], which then is used to derive explicit formulas for conformal maps of circular domains onto the canonical slit domains. The formula in the latter paper contains infinite products which converge for domains that are sufficiently separated. However, it is not known if they always converge.

Paul Koebe [12] introduced several of the canonical slit domains. There have been demonstrated deep connections between the Dirichlet and Neumann problems in multiply connected domains and conformal slit mappings, potential theory and extremal problems [14], [15], [16].

We define here the canonical slit domains presented in Nehari's book [14].

The slit disk domain

Let $\mathbb{D}$ denote the unit disk centered at the origin. These domains are obtained by removing finitely many arcs from $\mathbb{D}$. Each of these arcs must be an arc of a circle centered at the origin.

The slit annulus

These domains are obtained by removing finitely many arcs from an annulus whose outer circle is $\partial\mathbb{D}$. Again, each of these arcs must be an arc of a circle centered at the origin.

The circular slit domain

These domains are obtained by removing from $\hat{\mathbb{C}}$ one or more arcs. Again, each of these arcs must be an arc of a circle centered at the origin.

The radial slit domain

These domains are obtained by removing from $\hat{\mathbb{C}}$ one or more line segments which do not pass through the origin. Each of these line segments, when extended indefinitely in both directions, must yield a line that passes through the origin.

The parallel slit domain

These domains are obtained by removing from $\hat{\mathbb{C}}$ one or more parallel line segments.

We will first show that one can compute the conformal mappings onto a slit disk domain using a result of Max Schiffer [15]. We will then use the relations described in Nehari [14] between the conformal maps onto these domains to compute the conformal mappings onto the slit annulus and circular slit domains. We use the constructions in Schiffer's monograph to compute the maps onto the radial slit and parallel slit domain.

We will use Type-Two Effectivity [17] as our model of computation over spaces whose cardinality is that of the reals. The naming systems we will use are described in Section 3.1 of [2]. Since these are the only naming systems we will use, we will suppress their mention. We will also talk about computations on objects when it is clear that we are really talking about computations with names of objects. We will write our proofs in a fairly informal style. In particular, we will rely on the informal definitions in Section 3.1 or [2].

2 Background from complex and harmonic analysis

Let D be a Jordan domain with boundary curves $\Gamma_1, \ldots, \Gamma_n$. For each $z \in D$, define $\omega(z, \Gamma_j, D)$ to be the value at z of the solution to the Dirichlet problem with boundary data

$$f(\zeta) = \begin{cases} 1 & \zeta \in \Gamma_j \\ 0 & \text{otherwise} \end{cases}$$

The function ω is called *harmonic measure*.

The *normal derivative* of u is denoted $\frac{\partial u}{\partial n}$ and is defined to be

$$\left(\frac{\partial u}{\partial x} y'(t) - \frac{\partial u}{\partial y} x'(t)\right) \frac{1}{|x'(t) + iy'(t)|}$$

when (x, y) is a positively oriented smooth Jordan curve. In this case, we also define

$$\frac{\partial u}{\partial s} = \left(\frac{\partial u}{\partial x} x'(t) + \frac{\partial u}{\partial y} y'(t)\right) \frac{1}{|x'(t) + iy'(t)|}.$$

If v is a harmonic conjugate of u, then it follows from the Cauchy-Riemann equations that

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial s}, \text{ and}$$
$$\frac{\partial v}{\partial n} = -\frac{\partial u}{\partial s}.$$

If u is harmonic, and if γ is a boundary component of $\mathrm{dom}(u)$, then the *period of the conjugate of u about γ* is defined to be

$$\frac{1}{2\pi} \int_\gamma \frac{\partial u}{\partial n} |dz|$$

where $|dz| =_{df} |z'(t)| dt$ is the differential of arc length. To make sense of this integral, we first use Schwarz Reflection to extend the domain of u to an open set containing γ.

Let G_D denote the Green's function of domain D. The following well-known result will be useful and is Corollary II.2.6 of [7].

Proposition 1. *Suppose D is a Jordan domain with smooth boundary curves $\Gamma_1, \ldots, \Gamma_n$. Then,*

$$\omega(\zeta, \Gamma_j, D) = -\frac{1}{2\pi} \int_{\Gamma_j} \frac{\partial G_D(z,\zeta)}{\partial n_z} |dz|.$$

Suppose f is a conformal map of a domain D onto a domain D_1, and that γ, γ_1 are boundary components of D, D_1 respectively. Suppose that whenever $\{z_n\}_{n=0}^{\infty}$ is a sequence of points in D such that $\lim_{n\to\infty} d(z_n, \gamma) = 0$, $\lim_{n\to\infty} d(f(z_n), \gamma_1) = 0$. We say that f *maps* γ *to* γ_1.

We will follow the convention of identifying a curve with its parameterizations.

3 A summary of previous results

The following three results, which are Theorems 5.2, 5.5, and 6.2 of [2], form the cornerstone for our work. Intuitively, the first of these theorems states that differentiation of harmonic functions is a computable operation.

Theorem 1 (Computable differentiation of harmonic functions). *From a name of a harmonic function, u, we may compute a name of $u'|_{\mathbb{C}}$.*

The second of these results states, roughly speaking, that solving Dirichlet problems (*i.e.* finding a harmonic function on a Jordan domain from the knowledge of its values on the boundary of the domain) is a computable operation.

Theorem 2 (Computable Solution of Dirichlet Problems). *Given a name of a Jordan domain D and names of smooth $\gamma_1, \ldots, \gamma_n$ and their derivatives, if $\gamma_1, \ldots, \gamma_n$ are the distinct boundary components of D, and if we are also given a name of a continuous $f : \partial D \to \mathbb{R}$, then we can compute a solution of the corresponding Dirichlet problem. Furthermore, we can compute an extension of this solution to $\overline{D}$.*

The third of these results demonstrates the computability of a matrix (known as the *Riemann matrix*) whose components are the periods around the boundary components of the harmonic conjugates of the harmonic measure functions.

Theorem 3 (Computability of the Riemann Matrix). *Given the same initial data as in Theorem 2, we can compute a name of the period of the harmonic conjugate of $\omega(\cdot, \Gamma_i, D)$ around Γ_j.*

4 Single-valued and multi-valued harmonic conjugates

Suppose u is a harmonic function with domain D. If D is 1-connected, then a harmonic conjugate of u may be defined by the equation

$$v(\zeta) = \int_{\zeta_1}^{\zeta} \frac{\partial u}{\partial n} |dz| \tag{1}$$

If D is multiply connected, then the right side of (1) may depend on the path of integration. In this case, u is said to have a *multi-valued harmonic conjugate*. Otherwise, u is said to have a *single-valued harmonic conjugate*. It follows that if D is contained in the interior of one of its boundary components then u has a single-valued harmonic conjugate if and only if its period around every other boundary component is zero.

It is well-known that if D is finitely connected and bounded by smooth Jordan curves, one of which contains D in its interior, then one can add a unique linear combination of the harmonic measure functions of the boundary components of D to u and obtain a function with a single-valued harmonic conjugate. Our first goal is to show that this can be done effectively.

Lemma 1. *Given a name of a harmonic function u defined on a finitely connected domain, D, and names of the boundary components of D, $\gamma_1, \ldots, \gamma_n$, we may compute $b_1, \ldots, b_{n-1}$ such that*

$$u + \sum_{j=1}^{n-1} b_j \omega(\cdot, \gamma_j, D) \tag{2}$$

has a single-valued harmonic conjugate provided $\gamma_1, \ldots, \gamma_n$ are smooth Jordan curves, D is contained in the interior of γ_n, and we are also given names of $\gamma_1', \ldots, \gamma_n'$.

Proof. Let $R_{k,j}$ be the period of $\omega(\cdot, \gamma_j, D)$ about γ_k.

We first want to compute the period of the conjugate of u about each γ_k. Denote this period by p_k. To compute p_k, we want γ_k to be positively oriented. This can be checked by using the *winding number*

$$\int_{\gamma_k} \frac{1}{z - \zeta}.$$

We can effectively search for a rational rectangle R such that $\overline{R} \subseteq \mathbb{C} - \gamma_k$ on which this winding number is non-zero. If this value is positive, we can in addition discover a positive lower bound on it. If it is negative, then we can in addition discover a negative upper bound on it. In the former case, we know γ_k is positively oriented. Otherwise, it is negatively oriented in which case we can reparameterize it positively. Hence, we will assume without loss of generality that each γ_k is positively oriented.

Now, let $R_{k,j}$ be the period of $\omega(\cdot, \gamma_j, D)$ about γ_k. It is well-known that the matrix $(R_{k,j})_{k,j=1,\ldots,n-1}$ is invertible. (See, *e.g.*, Section I.10 of [14].) To ensure that the function in (2) has no conjugate period about γ_k, $k = 1, \ldots, n-1$, it suffices to show that

$$R_{k,1} b_1 + \ldots + R_{k,n-1} b_{n-1} = -p_k.$$

It now follows from the results in [18] that $b_1, \ldots, b_{n-1}$ can be computed from the given information.

5 The slit disk domain

The following lemma will be useful.

Lemma 2 (Conformal Reconfiguring Lemma). *Given a name of a non-degenerate domain D, a name of its boundary, and the number of its boundary components, we can compute a domain D_1, its boundary, a conformal map f of D onto D_1, and smooth Jordan curves $\gamma_1, \ldots, \gamma_n$ and their derivatives such that $\gamma_1, \ldots, \gamma_n$ are the boundary components of D_1. Furthermore, γ_n is a circle. Furthermore, if we are in addition given a name of a boundary component γ of D we can ensure that f maps γ to γ_n.*

Proof. Follow the first n steps of the Koebe Construction (for details see Section 2 and Theorem 4.6 of [2]): let $D_{0,1}, ..., D_{0,n}$ denote the connected components of the complement of D. At the first step with the help of a Riemann mapping, map the complement of $D_{0,1}$ conformally onto the unit disk $D'_{0,1}$. It follows from Theorem 5.1 of [9] that we can compute this map from the given data. The boundary δ_1 of $D_{0,1}$ is transformed into the unit circle δ'_1. $D_{0,2}$ is mapped onto $D'_{0,2}$, δ_2 into δ'_2 *etc.*. At step two map the complement of $D'_{0,2}$ onto the unit disk using a Riemann mapping. The image of δ'_1 under the second Riemann mapping is an analytic curve. After n analogous steps the images $\gamma_1, ..., \gamma_n$ of $\delta_1, ..., \delta_n$ are analytic curves and γ_n is a circle. The

Theorem 4. *Given a name of a finitely connected, non-degenerate domain D, a name of its boundary, a name of one of its boundary components, γ, a name of a point $\zeta_0 \in \mathbb{D}$, and the number of boundary components of D, we can compute a conformal mapping of D onto a slit disk domain that maps ζ_0 to 0 and γ to $\partial\mathbb{D}$.*

Proof. We first apply the Conformal Reconfiguring Lemma. Let $f, D_1, \gamma_1, \ldots, \gamma_n$ be thusly obtained. We may assume f maps γ onto γ_n. Let $\zeta_1 = f(\zeta_0)$. We can now compute the center and radius of γ_n. Label these ξ and R respectively. Let $D_2, \Gamma_1, \ldots, \Gamma_n, \zeta_2$ be the images of $D_1, \gamma_1, \ldots, \gamma_n, \zeta_1$ under the inversion map $z \mapsto \frac{R^2}{z-\xi}$.

Let G be the Green's function of D_2. It follows from Theorem 2 that we can compute G from the given information.

Let $\omega_j(z) = \omega(z, \Gamma_j, D_2)$. Compute $b_1(\zeta), \ldots, b_{n-1}(\zeta)$ as in the proof of Lemma 1 for the function $G(\cdot, \zeta)$. Let

$$m(z,\zeta) = G(z,\zeta) + \sum_{j=1}^{n-1} b_j(\zeta)\omega_j(\zeta).$$

It follows that m has no conjugate period about any of $\gamma_1, \ldots, \gamma_{n-1}$. A fairly straightforward calculation shows that m has a period of 1 about ζ. So, for all $z_0 \in D_2 - \{\zeta_2\}$, let

$$g(z_0) = \exp\left(-m(z,\zeta_2) - i\int_{\zeta_2}^{z_0} \frac{\partial m(z,\zeta_2)}{\partial n_z}|dz|\right).$$

Extend g to all of D_2 by setting $g(\zeta_2) = 0$. It follows that g is single-valued and analytic. Note that $g(\zeta_2) = 0$.

A fairly straightforward calculation shows that g is the function in (A1.21) of [15]. Hence, g is the conformal mapping of D_2 onto a slit disk domain Ω that maps ζ_2 to 0 and Γ_n to $\partial\mathbb{D}$.

It now only remains to show that we can compute a name of g from the given data. It suffices to show that from the given data and a name of a point $z \in D_2$ we can compute a name of $g(z)$ (see *e.g.* Theorem 3.3.15.2 of [17]).

When $z \neq \zeta_2$ (which, if true, will eventually be witnessed as we read the name of z), we can through an effective search procedure discover a piecewise linear path of integration contained in $D_2 - \{\zeta_2\}$. If z is in a subbbasic neighborhood of ζ_2 whose closure is contained in D_2, we can compute a positive lower bound on $G(z, \zeta_2)$ and arrive at a subbasic neighborhood of 0 which will contain $g(z)$. In either case, the computed neighborhoods will converge to $g(z)$ if the input neighborhoods converge to z.

It is worth noting that in the case when D is a circular domain, one can obtain explicit formulas for the slit-disk mapping function using the formulas for the Green's function and harmonic measure in [10].

6 Some immediate consequences of the slit disk result

Let 'SD' stand for 'slit disk', 'CS' for 'circular slit', *etc.*. We introduce some notation for the conformal maps onto these domains. Fix a non-degenerate, finitely connected domain D. Let $\zeta_0, \zeta_1 \in D$, and let $\gamma_1, \ldots, \gamma_n$ be the boundary components of D. We then let $f_{SD}(\cdot; D, \zeta_0, \gamma_j)$ denote the unique conformal map of D onto a slit disk domain that maps ζ_0 to 0 and γ_j onto $\partial\mathbb{D}$ whose derivative at ζ_0 is positive.

Let $f_{CS}(\cdot; D, \zeta_0, \zeta_1)$ be the conformal map of D onto a circular slit domain that maps ζ_0 to 0, ζ_1 to ∞, and whose residue at ζ_1 is 1.

Let $f_{PS}(\cdot; D, \zeta_1, \theta)$ be the conformal map of D onto a parallel slit domain where all slits have angle θ with the x-axis and whose Laurent expansion at ζ_1 is of the form

$$\frac{1}{z - \zeta_1} + a(z - \zeta_1) + b(z - \zeta_1)^2 + \ldots.$$

Let $f_{RS}(\cdot; D, \zeta_0, \zeta_1)$ be the conformal map of D onto a radial slit domain that maps ζ_0 to 0, ζ_1 to ∞, and whose residue at ζ_1 is 1.

Let $f_{SA}(\cdot; D, \gamma_j, \gamma_k)$ be the conformal map of D onto a slit annulus domain that maps γ_j onto $\partial\mathbb{D}$ and γ_k onto the inner circle.

We omit any of these parameters when they are made clear by context.

Suppose we are given a name of a finitely connected, non-degenerate domain D, a name of its boundary, and the number of its boundary components. It is now required to show that we can compute these other canonical maps uniformly in the parameters beyond the semicolon. In the case of f_{CS} and f_{SA}, this follows from the following identities which are proven in Section VII.1 of [14].

$$f_{CS}(z;\zeta_0,\zeta_1) = \frac{f'_{SD}(\zeta_1;\zeta_1)}{f'_{SD}(\zeta_1;\zeta_0)}\frac{f_{SD}(z;\zeta_0)}{f_{SD}(z;\zeta_1)}$$
$$f_{SA}(z;\gamma_j,\gamma_k) = \frac{f_{SD}(z;\zeta_0,\gamma_j)}{f_{SD}(z;\zeta_0,\gamma_k)}$$

We now discuss the computation of $f_{PS}(\cdot;D,\zeta,\theta)$. Let ζ_x, ζ_y denote the real and imaginary parts of ζ respectively. It is shown on page 256 of [15] that

$$f_{PS}(z;D,\zeta,\pi/2) = -\frac{\partial}{\partial\zeta_x}\log f_{SD}(z;D,\zeta,\gamma_j)$$
$$f_{PS}(z;D,\zeta,0) = -\frac{1}{i}\frac{\partial}{\partial\zeta_y}\log f_{SD}(z;D,\zeta,\gamma_j)$$

It then follows (as on page 257 of [15]) that

$$f_{PS}(z;D,\zeta,\theta) = e^{i\theta}[\cos(\theta)f_{PS}(z;D,\zeta,0) - i\sin(\theta)f_{PS}(z;D,\zeta,\pi/2)].$$

Hence, we may compute $f_{PS}(\cdot;D,\zeta,\theta)$ from the given data.

In order to compute the conformal mappings onto the other canonical domains, we make a digression and consider the Neuman problem.

7 Digression: computing solutions to the Neuman problem

Let D be a bounded domain with smooth boundary curves $\Gamma_1,\ldots,\Gamma_n$. Let $f \in C(\partial D)$, and suppose $\int_{\partial D} f|dz| = 0$. The resulting *Neuman problem* is to find a harmonic function u on D such that

$$\frac{\partial u}{\partial n} = f \text{ on } \partial D \tag{3}$$
$$\int_{\partial D} u|dz| = 0 \tag{4}$$

Such solutions exist (see, *e.g.* Appendix B of [7]). The condition 4 ensures they are unique. Their computability will now be demonstrated by a well-known procedure (see *e.g.* proof of Theorem B.1 in [7]).

Theorem 5 (Computing solutions of Neuman problems). *Given a name of a bounded domain D, names of n smooth Jordan curves $\Gamma_1,\ldots,\Gamma_n$ which form its boundary components as well as names of their derivatives, and a name of $f \in C(\partial D)$ such that (3) holds, one can compute a name of the solution of the resulting Neuman problem.*

Proof. There is already a well-known 'procedure' for finding solutions to Neuman problems. The purpose of this proof is to explain this procedure and show that it can be implemented on a digital computing device.

By using the winding number and a simple search procedure, we can determine which of $\Gamma_1, \ldots, \Gamma_n$ contains D in its interior. Without loss of generality, suppose Γ_n is this curve.

As in the proof of Lemma 1, we can assume $\Gamma_1, \ldots, \Gamma_n$ are positively oriented.

Let $R_{j,k}$ be the period of the conjugate of ω_k about Γ_j. As noted in the proof of Lemma 1, the matrix $(R_{j,k})_{j,k=1,\ldots,n-1}$ is invertible. So, we can now compute the solution to the system of linear equations

$$R_{j,1}b_1 + \ldots + R_{j,n-1}b_{n-1} = \int_{\Gamma_j} f|dz| \; j = 1, \ldots, n-1.$$

Let

$$f_1 = f - \sum_{k=1}^{n-1} b_k \frac{\partial \omega_k}{\partial n}.$$

It follows that $\int_{\Gamma_j} f_1|dz| = 0$ if $j \in \{1, \ldots, n-1\}$.

It is an easy consequence of Green's Theorem that $R_{j,k} = R_{k,j}$. (See, *e.g.*, Section I.10 of [14].) It is also easy to show that for each j, the sum of the periods of the harmonic conjugates of $\omega_1, \ldots, \omega_n$ is 0. (One first notes that the sum of the harmonic measure functions is identically 1 on ∂D.) Since $\int_\Gamma f|dz| = 0$, it now follows by a fairly straightforward calculation that $\int_{\Gamma_n} f_1|dz| = 0$.

We now wish to define a function g on ∂D. We do so by defining it on each boundary component of D. When $\zeta \in \Gamma_j$, we let

$$g(\zeta) = \int_0^{t_0} f_1(\Gamma_j(t))|\Gamma_j'(t)|dt$$

where t_0 is such that $\Gamma_j(t_0) = \zeta$. Since $\int_{\Gamma_j} f_1|dz| = 0$, it follows that the choice of t_0 is irrelevant when $\zeta = \Gamma_j(0)$. Hence, g is well-defined.

It is now necessary to prove the following Lemma.

Lemma 3. *g can be computed from the given data.*

Proof. Let $\omega_j = \omega(\cdot, \Gamma_j, D)$.

Suppose we are given a name of a point $\zeta \in \partial D$ as input. From our name for a parametrization of Γ_j, we can compute names of Γ_j as a closed subset of the plane as well as a name of the open set $\mathbb{C} - \Gamma_j$. (See, *e.g.*, Theorem 6.2.4.4 of [17].) We then scan these names and our name for ζ until we find a rational rectangle R and an index j such that $\zeta \in R$, $R \cap \Gamma_j \neq \emptyset$, and $\overline{R} \cap \Gamma_k = \emptyset$ when $k \neq j$. Hence, we now know $\zeta \in \Gamma_j$. Begin computing a name for the function h defined by

$$h(t) = \int_0^t f_1(\Gamma_j(t))|\Gamma_j'(t)|dt.$$

We now continue scanning our name for ζ and our generated names for $\Gamma_j(0)$ and h. Suppose that at some point in this process we discover disjoint rational rectangles R_1, R_2 such that $\zeta \in R_1$ and $\Gamma_j(0) \in R_2$. So, we now know $\zeta \neq \Gamma_j(0)$. Hence, there is exactly one value of t for which $\Gamma_j(t) = \zeta$ and we can compute this value. (See, *e.g.* Corollary 6.3.5 of [17].) Hence, we can now compute $g(\zeta)$ directly from the definition of g.

Suppose on the other hand that at some point in this process no such rational rectangles have been discovered. We then search the portions of these names read so far for R, $[a_1, b_1], \ldots, [a_m, b_m]$, $R_1, \ldots, R_m$, R', I_1, and I_2 such that

1. $\zeta, \Gamma_j(0) \in R$,
2. Γ_j maps $[a_l, b_l]$ into R_l,
3. $[a, b] =_{df} \bigcup_l [a_l, b_1]$ is a subinterval of $(0, 1)$,
4. $R_l \cap \overline{R} = \emptyset$,
5. h maps $[0, a]$ into I_1, and
6. h maps $[b, 1]$ into I_2.

If this search fails, then we continue scanning. If it succeeds, then, although we do not know yet if $\zeta = \Gamma_j(0)$, we do know that any Γ_j preimage of ζ lies in $[0, a] \cup [b, 1]$. Note that $0 \in I_1 \cap I_2$. So, we can list, for each successful search of this kind, $I_1 \cup I_2$ as an interval that contains $g(\zeta)$. We can also in the future interleave listing of all rational intervals that contain $I_1 \cup I_2$.

We now show that this process generates a name for $g(\zeta)$. Every interval listed contains $g(\zeta)$. So, we only need to show that every interval that contains $g(\zeta)$ is eventually listed. This is clearly true if $\zeta \neq \Gamma_j(0)$. Suppose $\zeta = \Gamma_j(0)$. It follows that there will be infinitely many successful search of the kind described above. It also follows that larger portions of these names are read, the diameter of $I_1 \cup I_2$ will tend to 0. It follows that a name of $0 = g(\Gamma_j(0))$ is written on the output tape.

We now compute the solution to the Dirichlet problem for D with boundary data g. Call this solution v_1. We now compute $a_1, \ldots, a_{n-1}$ such that

$$v =_{df} v_1 + \sum_{j=1}^{n-1} a_j \omega_j$$

has a single-valued harmonic conjugate. Note that since ω_j is constant on each curve of ∂D, $\frac{\partial v}{\partial s} = f_1$. Compute $\xi \in D$. Let:

$$u_1(z_0) = \int_{\xi}^{z_0} \frac{\partial(-v)}{\partial n} |dz|$$

$$u_2 = u_1 + \sum_{j=1}^{n-1} b_j \omega_j$$

Since u_1 is a harmonic conjugate of $-v$, it follows that the normal derivative of u_1 on ∂D is f_1. It now follows that f is the normal derivative of u_2 on ∂D. We now complete our computation by setting

$$u = u_2 - \int_{\Gamma} u_2 |dz|.$$

If D is a bounded domain bounded by smooth Jordan curves $\Gamma_1, \ldots, \Gamma_n$, then the *Neuman function* of D, N_D, is defined by the following conditions.

1. $z \mapsto N(z,\zeta) + \log|z-\zeta|$ is harmonic.
2. $\frac{\partial}{\partial n_z} N(z,\zeta) = -\frac{2\pi}{L}$ on ∂D where L is the length of ∂D.
3. $\int_{\partial D} N(z,\zeta)|dz| = 0$.

Corollary 1. *From names of D, $\Gamma_1, \ldots, \Gamma_n$, $\Gamma_1', \ldots, \Gamma_n'$ as in Theorem 5, we can compute a name of N_D.*

8 The radial slit domain

We conclude with the following.

Theorem 6. *From a name of non-degenerate, finitely connected, domain D, a name of its boundary, names of distinct $\zeta_0, \zeta_1 \in D$, and the number of its boundary components, we can compute a name of $f_{RS}(\cdot; D, \zeta_0, \zeta_1)$.*

Proof. Let $u(z) = N_D(z,\zeta_0) - N_D(z,\zeta_1)$. It follows that u has a single-valued harmonic conjugate. So, let

$$\tilde{u}(z_0) = \int_{\zeta_0}^{z_0} \frac{\partial u}{\partial n} |dz|$$

when $z_0 \neq \zeta_0, \zeta_1$. Let $f = \exp(-(u+i\tilde{u}))$. Extend f to all of D by setting $f(\zeta_0) = 0$ and $f(\zeta_1) = \infty$. It is shown in [15] (page 265, (A1.62)) that $f_{RS}(\cdot, D, \zeta_0, \zeta_1) = f$. It only remains to demonstrate that we can compute f from the given data.

Suppose we are given the name of $z \in D$ as input. Scan the names of z, ζ_0, ζ_1. If at some point, we discover disjoint rational rectangles R_1, R_2, R_3 such that $z \in R_1$, $\zeta_0 \in R_2$, and $\zeta_1 \in R_3$, then we can compute $u(z)$ and $\tilde{u}(z)$ directly. Suppose at some point we have not found such rectangles. If we have discovered a rational rectangle R that contains z and ζ_0 but not ζ_1, we can compute a positive lower bound on $N_D(z,\zeta_0) - N_D(z,\zeta_1)$ and hence a neighborhood of 0 that contains $f(z)$. If we have discovered a rational rectangle R that contains z and ζ_1 but not ζ_0, we can compute a negative upper bound on $N_D(z,\zeta_0) - N_D(z,\zeta_1)$ and hence a neighborhood of ∞ that contains $f(z)$. By continuing this process indefinitely, we generate a name of $f(z)$.

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